

INTERNAL ZONOTOPAL ALGEBRAS AND THE MONOMIAL REFLECTION GROUPS $G(m, 1, n)$

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ABSTRACT. The group $G(m, 1, n)$ consists of n -by- n monomial matrices whose entries are m th roots of unity. It is generated by n complex reflections acting on \mathbf{C}^n . The reflecting hyperplanes give rise to a (hyperplane) arrangement $\mathcal{G} \subset \mathbf{C}^n$. The internal zonotopal algebra of an arrangement is a finite dimensional algebra first studied by Holtz and Ron. Its dimension is the number of bases of the associated matroid with zero internal activity. In this paper we study the structure of the internal zonotopal algebra of the Gale dual of the reflection arrangement of $G(m, 1, n)$, as a representation of this group. Our main result is a formula for the top degree component as an induced character from the cyclic group generated by a Coxeter element. We also provide results on representation stability, a connection to the Whitehouse representation in type A, and an analog of decreasing trees in type B.

1. INTRODUCTION

A meta-problem in the theory of hyperplane arrangements starts with an algebraic object $M(\mathcal{A})$ derived from an arrangement \mathcal{A} and determines the extent to which the intersection lattice $L(\mathcal{A})$ determines $M(\mathcal{A})$ up to isomorphism. The prototypical example of this associates to an arrangement $\mathcal{A} \subset V$ the cohomology ring of the complement $H^*(V - \mathcal{A}; \mathbf{Z})$. The structure of this ring was determined by Orlik and Solomon [11] — it is determined by the combinatorics of no-broken-circuit subsets of the associated matroid.

There is an equivariant version of the meta-problem where the arrangement \mathcal{A} is fixed by the action of a group W acting linearly on V . When this happens one wants to determine the structure of $M(\mathcal{A})$ as a representation of W . The prototypical example here is that of a real reflection group W acting together with its reflection arrangement $\mathcal{A} \subset V$ and one wants to understand the structure of $H^*(V - \mathcal{A}; \mathbf{Z})$ as a representation of W . For complex reflection groups this problem first studied by Orlik and Solomon [12] and later by many others.

Of recent interest are the so-called zonotopal algebras of a hyperplane arrangement of Holtz and Ron [8]. These were rediscovered in many disparate areas of mathematics and unified under the guise of power ideals by Ardila and Postnikov in [1] (which contains an extensive list of references). The zonotopal ideals of an arrangement $\mathcal{A} \subset V$ are ideals in $Sym(V)$ generated by powers of linear forms. Specifically, the k th zonotopal ideal of \mathcal{A} is

$$\langle h^{\max\{\rho_{\mathcal{A}}(h)+k+1, 0\}} : h \in V \rangle$$

where $\rho_{\mathcal{A}}(h)$ is the number of hyperplanes in \mathcal{A} that do not contain h . The quotient $S_{\mathcal{A},k} = Sym(V)/I_{\mathcal{A},k}$ is the k th zonotopal algebra of \mathcal{A} . Holtz and Ron single out

the cases $k = -2, -1, 0$ as being of particular interest, and call these the internal, central and external zonotopal algebras of \mathcal{A} . The internal case $k = -2$ exhibits dramatic subtleties in comparison to all other cases $k > -1$ [2]. When $k \leq -6$ the zonotopal algebra no longer has structure determined by the lattice $L(\mathcal{A})$ [2, Proposition 4]. The intermediary cases have not been studied to our knowledge.

In this paper we study the internal zonotopal algebra of certain arrangements coming from complex reflection groups W . For numerical reasons we do not study the naturally occurring reflection arrangement of W , but instead its Gale dual. We are specifically interested in the structure of this algebra as representations of W . We summarize our main results here.

Theorem. *Let \mathcal{G} be the reflection arrangement of one of the monomial groups $G(m, 1, n) \subset GL_n(\mathbb{C})$, consisting of n -by- n permutation matrices whose entries are m th roots of unity ($m > 1$ and $n \geq 3$). Let \mathcal{G}^\perp denote the Gale dual of \mathcal{G} . Then the degree $n - 1$ component of $S_{\mathcal{G}^\perp, -2}$ is isomorphic to*

$$\mathrm{Ind}_C^W(e^{2\pi i(n-1)/mn})$$

as a representation of W , where $C \subset W$ is the cyclic subgroup generated by a Coxeter element.

We complement this result with several others on the structure of the internal zonotopal algebra $S_{\mathcal{G}^\perp, -2}$. We study the type A case of the above theorem, when $m = 1$. Based on our theorem above, it is perhaps unsurprising that the Lie representation appears in type A, however our perspective makes obvious the “hidden” action of a larger symmetric group than is necessary on the Lie representation, as observed by Mathieu [10] and Robinson and Whitehouse [14]. For all $G(m, 1, n)$, we give explicit generators of the internal zonotopal ideal and then use this description to prove finite generation in the sense of Sam and Snowden [16] and representation stability as described by Gan and Li [7]. We generalize a recurrence relation for the Whitehouse representation which factorizes the regular representation of $G(m, 1, n)$, $m > 1$. In type B, when $m = 2$, we use a Gröbner basis for the internal zonotopal ideal to compute an analog of decreasing trees, which are classic combinatorial objects.

2. ZONOTOPAL ALGEBRAS

Let \mathcal{A} be a central arrangement of hyperplanes in a complex finite dimensional vector space V . We let $M = M(\mathcal{A})$ denote the matroid of \mathcal{A} . Recall that this is the simplicial complex on \mathcal{A} whose faces (alias independent sets) are collections of hyperplanes whose defining linear forms are linearly independent. Maximal faces in M are referred to as its bases.

Define a function $\rho_{\mathcal{A}} : V \rightarrow \mathbb{N}$ by the rule

$$\rho_{\mathcal{A}}(h) = \text{the number of hyperplanes in } \mathcal{A} \text{ not containing } h,$$

and use this to define the ideal

$$I_{\mathcal{A}, k} = \langle h^{\max\{\rho_{\mathcal{A}}(h)+k+1, 0\}} : h \in V \rangle \subset \mathrm{Sym}(V).$$

For $k \geq 2$ the quotient $S_{\mathcal{A}, k} := \mathrm{Sym}(V)/I_{\mathcal{A}, k}$ has Krull dimension zero, and so it is a finite dimensional complex vector space. The vector space dimension of this quotient is the *degree* of the quotient as a $\mathrm{Sym}(V)$ module. The ring $\mathrm{Sym}(V)$ is

graded in the usual way, and we denote the m th graded piece of a graded module over this ring by $(-)_m$. The ideal $I_{\mathcal{A},k}$ is graded, since it is generated by powers of linear forms, which are homogeneous. The *Hilbert series* of $S_{\mathcal{A},k}$ is the generating function for the dimensions of the graded pieces of this graded $\text{Sym}(V)$ -module. Since $(S_{\mathcal{A},k})_m = 0$ for m sufficiently large there is a least m for which $(S_{\mathcal{A},k})_m \neq 0$ and we call this the *top* of quotient and denote it by $S_{\mathcal{A},k}(\text{top})$ to ease the proliferation of superscripts and subscripts.

2.1. Hilbert series. We will describe the Hilbert series of $S_{\mathcal{A},k}$ in terms of the matroid of \mathcal{A} . To do so, we need the Tutte polynomial of a matroid. We take the most expedient route. Given a matroid M with ground set E , the *rank* of a subset S of E , is size of the largest independent set of M contained in S . The *Tutte polynomial* of a matroid M with ground set E and rank function $\text{rk} : 2^E \rightarrow \mathbf{N}$ is the bivariate polynomial

$$T_M(x, y) = \sum_{A \subseteq E} (x-1)^{\text{rk}(E)-\text{rk}(A)} (y-1)^{|A|-\text{rk}(A)}.$$

The Tutte polynomial of a matroid is universal in the sense that any matroid invariant taking values in an abelian group satisfying a generalized deletion-contraction identity must be an evaluation of the Tutte polynomial.

For $k \geq -2$ there is a short exact sequence relating $S_{\mathcal{A},k}$ to $S_{\mathcal{A} \setminus H, k}$ and $S_{\mathcal{A}/H, k}$, where $\mathcal{A} \setminus H$ is \mathcal{A} with one of its defining hyperplanes H removed, and \mathcal{A}/H is the arrangement obtained by projecting \mathcal{A} into H . When interpreted at the level of Hilbert series this becomes a deletion-contraction relation and we have the following result.

Theorem 2.1 (Ardila–Postnikov [1], Holtz–Ron [8]). *Let \mathcal{A} be a central, essential arrangement of hyperplanes in V . Let $M = M(\mathcal{A})$ denote the matroid of the arrangement \mathcal{A} , which consists of m hyperplanes. The Hilbert series of $S_{\mathcal{A},k}$ is equal to*

- (1) $q^{m-\text{rk}(M)} T_M(1+q, 1/q)$ if $k = 0$;
- (2) $q^{m-\text{rk}(M)} T_M(1, 1/q)$ if $k = -1$;
- (3) $q^{m-\text{rk}(M)} T_M(0, 1/q)$ if $k = -2$.

The algebras $S_{\mathcal{A},k}$ occurring in the theorem are, respectively, referred to as the external, central and internal zonotopal algebras of \mathcal{A} by Holtz and Ron. The most subtle case of the theorem is the third [2, 8]. In the external and central cases, explicit bases of the *Macaulay inverse system* of $I_{\mathcal{A},k}$ can be given [1] in terms of independent sets of the matroid M with a given external activity. In the internal case, the analogous results fail [2] and there is no known canonical basis described by the matroid of \mathcal{A} . This is particularly striking for the top degree component of $S_{\mathcal{A},k}$, whose dimension is $T_M(0, 1)$ — the number of bases of M with internal activity zero. The number $T_M(0, 1)$ is also the reduced Euler characteristic of the independent set complex of M . Finding a combinatorial basis for $S_{\mathcal{A},-2}$ or its Macaulay inverse system is a tantalizing open problem.

A smaller generating set for $I_{\mathcal{A},k}$ suffices than the one given when $k \in \{-2, -1, 0\}$. To describe the smaller generating set, we recall that a *line* of the arrangement \mathcal{A} is a one-dimensional intersection of hyperplanes in \mathcal{A} .

Theorem 2.2 (Ardila–Postnikov [1]). *Let \mathcal{A} be a central, essential arrangement of hyperplanes in V . The ideal*

$$I'_{\mathcal{A},k} = \langle h^{\rho_{\mathcal{A}}(h)+k+1} : h \text{ is a line of } \mathcal{A} \rangle$$

is equal to $I_{\mathcal{A},k}$ for $k \in \{-2, -1, 0\}$.

3. COMPLEX REFLECTION GROUPS

A comprehensive treatment of reflection groups in unitary spaces is Lehrer and Taylor’s book. Let V be a complex vector space. A (generalized) reflection is an element of $GL(V)$ that fixes a hyperplane point-wise and has finite order.

A complex reflection group is a finite subgroup of $GL(V)$ generated by reflections. Such groups have been classified by Shephard and Todd, and Serre and Chevellay. There appears a single infinite family of groups $G(me, e, n)$, $m, e, n \geq 1$, called the monomial groups as well as 34 exceptional groups. The monomial group $G(me, e, n)$ consists of n -by- n permutation matrices whose entries are me -th roots of unit, the product of which is a d -th root of unity.

Associated to a complex reflection group W is its reflection arrangement $\mathcal{A} \subset V$. This is the arrangement of hyperplanes that are fixed by the reflections generating W .

The reflection arrangement $\mathcal{G} = \mathcal{G}_{m,1,n}$ associated to the group $G(m, 1, n)$ ($m > 1$) is defined in $V = \mathbf{C}^n$ by the coordinate hyperplanes $x_i = 0$ ($1 \leq i \leq n$) and the hyperplanes

$$x_i - e^{2\pi i k/m} x_j \quad (1 \leq i < j \leq n, 1 \leq k \leq m).$$

We will deal with the case $m = 1$ in Section 6.

3.1. Invariants and degrees. The Chevalley–Shephard–Todd theorem characterizes complex reflection groups as those subgroups $W \subset GL(V)$ for which the ring of polynomial invariants $Sym(V^*)^W$ is itself a polynomial ring. Algebraically independent generators of the invariant ring will not be unique, but their degrees $d_1 \leq d_2 \leq \dots \leq d_\ell$ will be and are referred to as the *degrees of W* . For example, the degrees of $G(m, 1, n)$ are $m, 2m, 3m, \dots, mn$.

The Coxeter number of W is the largest degree of W and will be denoted by h . For reflection groups $W \subset GL(V)$ generated by $\dim(V)$ reflections (which are called well-generated) the integers d'_i satisfying $d_i + d'_{n-i+1} = h$ are called the *codegrees of W* . The codegrees of $G(m, 1, n)$ are thus $0, m, 2m, \dots, (n-1)m$.

The Tutte polynomial evaluation $\chi_{\mathcal{A}}(q) = (-1)^{\text{rk}(\mathcal{A})} T_{\mathcal{A}}(1-q, 0)$ is called the characteristic polynomial of the arrangement. The polynomial $(-q)^{\text{rk}(\mathcal{A})} \chi_{\mathcal{A}}(-1/q)$ is Hilbert series of the cohomology ring of the complement of the arrangement $V - \mathcal{A}$. When \mathcal{A} is a reflection arrangement the relationship to the codegrees of W is this.

Theorem 3.1 (Orlik–Solomon [12, Theorem 5.5]). *Let \mathcal{A} be the reflection arrangement of a well-generated complex reflection group W with codegrees d'_1, \dots, d'_n . Then,*

$$q^{\text{rk}(\mathcal{A})} T_{\mathcal{A}}(1 + 1/q, 0) = \prod_{i=1}^n (1 + (1 + d'_i)q)$$

is the Hilbert series of $H^(V - \mathcal{A}; \mathbf{Z})$.*

3.2. Coxeter elements. A Coxeter element of W is an element $c \in W$ where

- (1) c has an eigenvector v that is not contained in any reflecting hyperplane of W ,
- (2) W acts freely on the orbit of v and,
- (3) the multiplicative order of the eigenvalue corresponding to v is the Coxeter number h .

It is shown in [13, Theorem 1.3] that when W is irreducible and well-generated by n reflections S , and (W, S) is a Coxeter system, being a Coxeter element is equivalent to being the product of the elements of S in any order.

In the group $G(m, 1, n)$, n generating reflections are given by the reflections that exchange the basis vectors x_i and x_{i+1} ($1 \leq i \leq n-1$) together with the generalized reflection that scales x_1 by $e^{2\pi i/m}$. It follows that a Coxeter element of $G(m, 1, n)$ is given by the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ e^{2\pi i/m} & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

This generates a cyclic subgroup of $G(m, 1, n)$ of order mn . When our arguments in Section 5 use a particular Coxeter element, we will use the one displayed here. The modifications required for a different Coxeter element will be evident.

4. GALE DUALITY

4.1. Definition and properties. Any arrangement of hyperplanes has a dual arrangement, determined as follows. If $\mathcal{A} \subset V$ is an arrangement then we can construct $\mathbf{C}^{\mathcal{A}}$, which is the vector space whose basis is in bijection with the hyperplanes defining \mathcal{A} .

Any choice of linear functionals defining \mathcal{A} determines a linear map $\mathbf{C}^{\mathcal{A}} \rightarrow V^*$. The kernel of this map is denoted K , so we have an exact sequence

$$0 \rightarrow K \rightarrow \mathbf{C}^{\mathcal{A}} \rightarrow V^*.$$

Dualizing gives an exact sequence

$$0 \leftarrow K^* \leftarrow (\mathbf{C}^{\mathcal{A}})^* \leftarrow V.$$

Since there is a natural basis of $(\mathbf{C}^{\mathcal{A}})^*$ indexed by the hyperplanes in \mathcal{A} , the map $(\mathbf{C}^{\mathcal{A}})^* \rightarrow K^*$ determines an arrangement in K . This is the *Gale dual* of \mathcal{A} , which we denote by \mathcal{A}^\perp . Assuming that the map to V^* above is surjective, Gale duality is an honest duality in the sense that $(\mathcal{A}^\perp)^\perp = \mathcal{A}$.

Gale duality can be interpreted at the level of matroids. Let M be a matroid whose ground set we denote by E and whose bases are denoted $\mathcal{B}(M)$. The *dual matroid* M^\perp of M has ground set E and

$$\mathcal{B}(M^\perp) = \{E - B : B \in \mathcal{B}(M)\}.$$

If M is the matroid associated to an arrangement \mathcal{A} then M^\perp is the matroid associated to \mathcal{A}^\perp .

Proposition 4.1. *Let M be a matroid and M^\perp its dual matroid. Then the Tutte polynomials of M and M^\perp are related by*

$$T_{M^\perp}(x, y) = T_M(y, x).$$

We employ this result to compute Hilbert series as such.

Proposition 4.2. *Let $\mathcal{A} \subset V$ be the reflection arrangement of well-generated complex reflection group W . Say that the coexponents of W are d'_1, \dots, d'_n . Then the Hilbert series of $S_{\mathcal{A}^\perp, -2}$ is*

$$\prod_{i=1}^n (1 + d'_i q)$$

In particular, the dimension of the degree $n - 1$ piece of $S_{\mathcal{A}^\perp, -2}$ is the product of the non-zero codegrees of W .

Proof. It follows from Theorem 2.1 that the Hilbert series in question is the Tutte polynomial evaluation $q^n T_{\mathcal{A}^\perp}(0, 1/q)$. By Proposition 4.1 this is $q^n T_{\mathcal{A}}(1/q, 0)$ and by the Theorem 3.1 we obtain

$$q^n T_{\mathcal{A}}(1/q, 0) = \prod_{i=1}^n (1 + d'_i q). \quad \square$$

Corollary 4.3. *Let \mathcal{G} be the reflection arrangement of the group $G(m, 1, n)$ and let \mathcal{G}^\perp be its dual. Then the Hilbert series of $S_{\mathcal{G}^\perp, -2}$ is*

$$(1 + mq)(1 + 2mq) \dots (1 + (n - 1)mq).$$

4.2. Group actions and duality. Assume that V is a representation of a group W and that W fixes \mathcal{A} in the sense that for any $w \in W$, $w\mathcal{A} = \mathcal{A}$. The arrangement \mathcal{A}^\perp can be made to carry an action of W in a natural way, as we now explain. Using the natural action of W on V^* , the elements of W act by permuting and rescaling of the linear functionals defining \mathcal{A} . This defines an action of W on $\mathbf{C}^{\mathcal{A}}$, whereby W acts by generalized permutation matrices. This action is cooked up so that the map $\mathbf{C}^{\mathcal{A}} \rightarrow V^*$ is equivariant.

Since $K \subset \mathbf{C}^{\mathcal{A}}$ is the kernel of a map of W -modules it is W invariant. Since the map $(\mathbf{C}^{\mathcal{A}})^* \rightarrow K^*$ is a map of W -modules we see that the arrangement \mathcal{A}^\perp is fixed in K by the natural action of W .

4.3. The Gale dual of the reflection arrangement of $G(m, 1, n)$. Let $\mathcal{G} \subset \mathbf{C}^n$ denote the reflection arrangement associated to the group $G(m, 1, n)$. We continue to assume that $m > 1$ until Section 6. In this section we give an explicit coordinatization of the Gale dual \mathcal{G}^\perp and describe how $W = G(m, 1, n)$ acts on the dual space K in which \mathcal{G}^\perp lives.

We begin by labeling the hyperplanes that define \mathcal{G} . To ease the notation (only slightly) we will write ω for $e^{2\pi i/m}$. Let h_i denote the hyperplane defined by the vanishing of the i -th coordinate function $e_i = 0$ in \mathbf{C}^n . Let h_{ij}^k denote the hyperplane defined by $e_i - \omega^k e_j$. We will use the convention that the superscript in this notation is read modulo m and that

$$h_{ji}^k = -\omega^k h_{ij}^{-k}.$$

The h 's form the basis for the vector space $\mathbf{C}^{\mathcal{G}}$ and the linear map $\mathbf{C}^{\mathcal{G}} \rightarrow \mathbf{C}^n$ sends $h_i \mapsto e_i$ and $h_{ij}^k \mapsto e_i - \omega^k e_j$.

The group W is generated by the n -by- n permutation matrices, which we identify with the symmetric group S_n , and the diagonal matrices g_i that have all diagonal entries 1 except the i -th which is ω . The action of W on $\mathbf{C}^{\mathcal{G}}$ is described in the following proposition.

Proposition 4.4. *Given $\pi \in S_n \subset W$,*

$$\pi h_i = h_{\pi(i)}, \quad \pi h_{ij}^k = h_{\pi(i)\pi(j)}^k.$$

For all i, j ,

$$g_i h_i = \omega h_i, \quad g_i h_{ji}^k = \omega h_{ij}^{k-1}, \quad g_j h_{ij}^k = h_{ij}^{k+1}.$$

If $\ell \notin \{i, j\}$ then $g_\ell h_{ij}^k = h_{ij}^k$.

The kernel K of the map $\mathbf{C}^{\mathcal{G}} \rightarrow \mathbf{C}^n$ is $m \binom{n}{2}$ dimensional and has basis given by

$$y_{ij}^k = (h_i - \omega^k h_j) - h_{ij}^k.$$

The arrangement $\mathcal{G}^\perp \subset K$ comes from restricting the linear functionals dual to the h 's to K . The action of W on K is immediate from the previous proposition.

Proposition 4.5. *Given $\pi \in S_n \subset W$,*

$$\pi y_{ij}^k = y_{\pi(i)\pi(j)}^k.$$

For all i, j ,

$$g_i y_{ij}^k = \omega y_{ij}^{k-1}, \quad g_j y_{ij}^k = y_{ij}^{k+1}.$$

If $\ell \notin \{i, j\}$ then $g_\ell y_{ij}^k = y_{ij}^k$.

An elementary computation shows the following result.

Proposition 4.6. *Let $c \in W$ denote the Coxeter element $g_1 \cdot (12 \dots n)$, which has order mn . Then*

$$c y_{(n-1)n}^0 = y_{n1}^1, \quad c y_{n1}^1 = \omega y_{12}^0.$$

5. THE MAIN THEOREM

Our goal in this section is to prove the following result.

Theorem 5.1. *Let \mathcal{G} be the reflection arrangement associated to the group $W = G(m, 1, n)$. Let $c \in W$ be a Coxeter element. There is an isomorphism of representations,*

$$S_{\mathcal{G}^\perp, -2}(\text{top}) \approx \text{Ind}_{\langle c \rangle}^W(e^{2\pi i(n-1)/mn}).$$

We require a lemma.

Lemma 5.2. *Let K be the kernel of the surjective linear map $\mathbf{C}^{\mathcal{G}} \rightarrow \mathbf{C}^n$.*

- (1) *K has a c eigenvector Y with eigenvalue $e^{2\pi i/mn}$.*
- (2) *The eigenvector Y cyclically generates K as a W -module.*

Proof of (1). Let $\xi = e^{-2\pi i/(mn)}$, so that $\xi^n = \omega^{-1}$ (recall that $\omega = e^{2\pi i/m}$). Define

$$Y := y_{12}^0 + \xi y_{23}^0 + \xi^2 y_{34}^0 + \cdots + \xi^{n-2} y_{(n-1)n}^0 + \xi^{n-1} y_{n1}^1.$$

(Notice that if $m = 1$ then this sum reduces to

$$y_{12}^0 + \xi y_{23}^0 + \xi^2 y_{34}^0 + \cdots + \xi^{n-2} y_{(n-1)n}^0 + \xi^{n-1} y_{n1}^0.)$$

Using the rules set forth in Proposition 4.5, in particular Proposition 4.6, we see that ξc acts on Y by

$$\xi y_{23}^0 + \xi^2 y_{34}^0 + \cdots + \xi^{n-1} y_{n1}^1 + \xi^n \omega y_{12}^0 = Y. \quad \square$$

Proof of (2). Assume that $n > 3$ and then consider $Y' := Y - (23)Y$:

$$Y' = (y_{12}^0 - y_{13}^0) + \xi(y_{23}^0 - y_{32}^0) + \xi^2(y_{34}^0 - y_{24}^0).$$

Let Z denote the \mathfrak{S}_n -module that Y' generates.

Consider the \mathfrak{S}_n -invariant subspace of K with basis y_{ij}^0 , $1 \leq i \neq j \leq n$. This module is isomorphic to $\bigwedge^2(\mathbf{C}^n)$ and hence there is an injection $Z \hookrightarrow \bigwedge^2(\mathbf{C}^n)$. The irreducible decomposition of $\bigwedge^2(\mathbf{C}^n)$ is $\bigwedge^2(\mathbf{1}^\perp) \oplus \mathbf{1}^\perp$, where $\mathbf{1}^\perp$ is the subspace of \mathbf{C}^n whose coordinates sum to zero [5, Exercise 4.6]. It will suffice to see that Y' does not lay entirely within one of these summands, since this implies that injection above is an isomorphism. It will follow that $y_{12}^0 \in Z$ and since y_{12}^0 generates K as a W -module, we will be done.

In the \mathfrak{S}_n -module \mathbf{C}^n define $z_i = e_i - e_{i+1}$, so that z_1, \dots, z_{n-1} forms a basis for $\mathbf{1}^\perp$ and $z_n = -(z_1 + \cdots + z_{n-1})$. Using the natural isomorphisms

$$\text{span}\{y_{ij}^0 : 1 \leq i < j \leq n\} \approx \bigwedge^2(\mathbf{C}^n) = \bigwedge^2(\mathbf{1}^\perp) \oplus \mathbf{1}^\perp,$$

we see that y_{ij}^0 maps to $(z_i \wedge z_j, z_i - z_j)$. Hence Y' maps to

$$(1) \quad (z_1 \wedge z_2 - z_1 \wedge z_3, -z_2 + z_3) + 2\xi(z_2 \wedge z_3, z_2 - z_3) + \xi^2(z_3 \wedge z_4 - z_2 \wedge z_4, z_3 - z_2).$$

The first component of (1) is

$$z_1 \wedge z_2 - z_1 \wedge z_3 + 2\xi(z_2 \wedge z_3) + \xi^2(z_3 \wedge z_4 - z_2 \wedge z_4).$$

The coefficient of $z_2 \wedge z_3$ (in the usual basis of $z_i \wedge z_j$, $1 \leq i < j \leq n$) is 2ξ if $n > 4$. If $n = 4$ then the coefficient of $z_2 \wedge z_3$ is $2\xi + 2$. In either case, the first component of (1) is not zero. The second component of (1) is

$$(-1 + 2\xi - \xi^2)(z_2 - z_3)$$

and this is not zero.

We now consider the case $n = 3$ and m even, where $Y = y_{12}^0 + \xi y_{23}^0 + \xi^2 y_{31}^1$. Consider the elements,

$$g_1^i Y = \omega^i y_{12}^i + \xi y_{23}^0 + \xi^2 y_{31}^{i+1}$$

for various values of i , and the \mathfrak{S}_3 modules they generate. Using the rule $y_{ij}^k = -\omega^k y_{ji}^{-k}$ we can see that modules generated by $g_1^i Y$, $0 \leq i < m/2$ have zero intersection since the largest superscript of a term y_{ij}^c , occuring in $g_1^i Y$ is $i + 1$. The \mathfrak{S}_3 module generated by each $g_i Y$ is isomorphic to $\mathbf{C}\mathfrak{S}_3$, for a $6m/2 = 3m$ dimensional subspace of K , which is all of K . It follows that the W orbit of Y is all of K .

When $n = 3$ and m is odd, we replace Y with a different eigenvector for the Coxeter element: $Y' = (1 + \xi c + \xi c^2)y_{12}^{(m-1)/2}$, which we can write as $Y' = y_{12}^{(m-1)/2} + \xi y_{23}^0 + \xi^2 \omega^{(m+1)/2} y_{13}^{(m-1)/2}$. Now apply $1 - (23) \in \mathbf{CW}$ to this element and get

$$(1 - \xi^2 \omega^{(m+1)/2})(y_{12}^{(m-1)/2} + y_{13}^{(m-1)/2})$$

Apply g_1 to this and divide by a scalar to get

$$y_{12}^{(m+1)/2} + y_{31}^{(m+1)/2} = -\omega^{(m+1)/2}(y_{21}^{(m-1)/2} + y_{31}^{(m-1)/2}).$$

The \mathfrak{S}_3 representation generated by $y_{21}^{(m-1)/2} + y_{31}^{(m-1)/2}$ and $y_{12}^{(m-1)/2} + y_{13}^{(m-1)/2}$ is 6 dimensional, and is thus equal to the representation generated by $y_{12}^{(m-1)/2}$, since the latter is 6 dimensional and contains the former. Now the W orbit of this variable spans K and we are done. \square

We use the previous result to prove that each graded piece of $S_{\mathcal{G}^\perp, -2}$ is a cyclic W -module.

Proposition 5.3. *For each $\ell \geq 0$, the W orbit of Y^ℓ spans the degree ℓ component of $S_{\mathcal{G}^\perp, -2}$.*

Proof. It is well known that $\text{Sym}^\ell(K)$ is spanned by ℓ th powers of linear forms. Since the W orbit of Y is K , the W orbit of $Y^\ell \in \text{Sym}^\ell(K)$ is all of $\text{Sym}^\ell(K)$. The corollary follows by taking the quotient. \square

Proof of Theorem 5.1. We know that $S_{\mathcal{G}^\perp, -2}(\text{top})$ contains a c eigenvector with eigenvalue $\xi^{-(n-1)} = e^{2\pi i(n-1)/mn}$, namely Y^{n-1} . It follows that there is a surjective map from $\text{Ind}_{\langle c \rangle}^W(e^{2\pi i(n-1)/mn})$ to the smallest W -module in $S_{\mathcal{G}, -2}(\text{top})$ containing Y^{n-1} . By Proposition 5.3 there is a surjection from $\text{Ind}_{\langle c \rangle}^W(e^{2\pi i(n-1)/mn})$ to $S_{\mathcal{G}, -2}(\text{top})$.

The dimension of the induced character is $|W|/|\langle c \rangle| = m^n n! / mn = m^{n-1}(n-1)!$ and by Corollary 4.3 this is the dimension of $S_{\mathcal{G}, -2}(\text{top})$. It follows that the surjection above is an isomorphism. \square

The group $W = G(m, 1, n)$ contains $G(m, 1, n-1)$ as a subgroup, by taking the direct sum of a matrix in $G(m, 1, n-1)$ with a 1-by-1 identity matrix.

Corollary 5.4. *As a representation of $G(m, 1, n-1)$, $S_{\mathcal{G}^\perp, -2}(\text{top})$ is isomorphic to the regular representation. That is,*

$$\text{Res}_{G(m, 1, n-1)}^W S_{\mathcal{G}^\perp, -2}(\text{top}) \approx \mathbf{C}[G(m, 1, n-1)].$$

Proof. The fact that the character of $S_{\mathcal{G}^\perp, -2}(\text{top})$ is an induced character allows us to see that it is zero at any element of $G(m, 1, n-1) \subset W$, except the identity where it is $|G(m, 1, n-1)|$. The result follows. \square

6. TYPE A

In this section we investigate the case when $m = 1$ in $G(m, 1, n)$, so that our reflection group is the symmetric group of n -by- n permutation matrices. To emphasize the dependence on n here we will write \mathfrak{S}_n instead of W .

The reflection arrangement of \mathfrak{S}_n is the well-studied braid arrangement $\mathcal{A} \subset \mathbf{C}^n$ whose defining hyperplanes are

$$x_j - x_i, \quad (1 \leq i < j \leq n).$$

The Macaulay inverse system of the central zonotopal ideal $I_{\mathcal{A}, -1}$ has dimension n^{n-2} , and was studied by the author and Rhoades in [3]. There it was shown to be a representation of the symmetric group \mathfrak{S}_n that restricted to the well-studied *parking representation* of \mathfrak{S}_{n-1} . This is the representation with basis give by sequences $\mathbf{p} = (p_1, \dots, p_{n-1})$ whose non-decreasing rearrangement \mathbf{q} satisfies $q_j \leq j$ for all $1 \leq j \leq n-1$.

To identify the Gale dual of \mathcal{A} we label the hyperplane $x_i - x_j = 0$ by h_{ij} . Those h_{ij} with $1 \leq i < j \leq n$ form a basis for $\mathbf{C}^{\mathcal{A}}$, and we stipulate that $h_{ji} = -h_{ij}$. The kernel K of the natural map $\mathbf{C}^{\mathcal{A}} \rightarrow \mathbf{C}^n$ is $\binom{n}{2} - n = \binom{n-1}{2}$ dimensional, and a basis is given by

$$y_{ij} = h_{ij} - h_{in} - h_{jn} \quad (1 \leq i < j \leq n-1).$$

One can check that $y_{ji} = -y_{ij}$. This kernel can be naturally identified with the cycle space of the complete graph on n vertices, which is spanned by the characteristic vectors of length 3 oriented cycles. A basis is then given by those 3 cycles that visit the vertex labeled n .

The transposition $(kn) \in \mathfrak{S}_n$ acts on the variables y_{ij} by the rule,

$$(2) \quad (kn)y_{ij} = \begin{cases} y_{ij} + y_{jk} + y_{ki}, & \text{if } k \notin \{i, j\}, \\ -y_{ij} & \text{if } k \in \{i, j\}. \end{cases}$$

Notice that the action of $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ on K agrees with action set forth in Section 4.3 with $m = 1$ and n replaced by $n-1$. We have the following analog of Theorem 5.1.

Theorem 6.1. *Let \mathcal{A} denote the reflection arrangement of \mathfrak{S}_n . Let $c \in \mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ denote a Coxeter element, which is an $(n-1)$ -cycle. As a representation of \mathfrak{S}_{n-1} , $S_{\mathcal{A}^\perp, -2}(\text{top})$ is isomorphic to $\text{Ind}_{\langle c \rangle}^{\mathfrak{S}_{n-1}}(e^{2\pi i/(n-1)})$.*

As a representation of $\mathfrak{S}_{n-2} \subset \mathfrak{S}_{n-1}$, $S_{\mathcal{A}^\perp, -2}(\text{top})$ is isomorphic to the regular representation.

The induced representation here $\text{Ind}_{\langle c \rangle}^{\mathfrak{S}_{n-1}}(e^{2\pi i/(n-1)})$ is called the Lie representation, which we write as Lie_{n-1} . It is the representation of \mathfrak{S}_{n-1} afforded by the multilinear component of the free Lie algebra on $n-1$ letters. It was shown by Stanley [cite] that the tensor product of the Lie representation with the sign character is isomorphic to the top cohomology of the partition lattice, and this is isomorphic to the top cohomology of the complement of the braid arrangement $\mathbf{C}^n - \mathcal{A}$.

Proof. The proof is the same as for Theorem 5.1, except that we need to know that $\dim S_{\mathcal{A}^\perp, -2}(\text{top})$ is $(n-2)!$. This follows as it did before because \mathcal{A} is a free arrangement with exponents $1, 2, \dots, n-1$ which means that the Hilbert series of $S_{\mathcal{A}^\perp, -2}(\text{top})$ is $(1+q)(1+2q) \cdots (1+(n-2)q)$. \square

6.1. The Whitehouse representation and the ideal of $S_{\mathcal{A}^\perp, -2}$. The structure of $S_{\mathcal{A}^\perp, -2}(\text{top})$ as a representation of \mathfrak{S}_n is more subtle, but turns out to have been computed before in work of Mathieu [10] and Gaiffi [6]. To state this result we need

to define the Whitehouse representation of the symmetric group \mathfrak{S}_n . This is the (*a priori* virtual) representation,

$$Wh_n := \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n}(\text{Lie}_{n-1}) - \text{Lie}_n,$$

which was first studied by Kontsevich in the context of free Lie algebras and later by Robinson and Whitehouse [14].

Theorem 6.2. *There is an isomorphism of representations of \mathfrak{S}_n , $S_{\mathcal{A}^+,-2}(\text{top}) \approx Wh_n$.*

To prove this we compute a generating set for the defining ideal of $S_{\mathcal{A}^+,-2}$. This is an ideal in the polynomial ring $\text{Sym}(K) = \mathbb{C}[y_{ij} : 1 \leq i < j \leq n-1]$.

Lemma 6.3. *The ideal $I_{\mathcal{A}^+,-2}$ is equal to*

$$J = \langle y_{ij}^2 : 1 \leq i < j \leq n \rangle + \langle y_{ij}y_{ki} + y_{ji}y_{kj} + y_{ki}y_{jk} : 1 \leq i < j < k \leq n \rangle$$

We start by proving one containment.

Proposition 6.4. *The ideal $I = I_{\mathcal{A}^+,-2}$ contains J .*

Proof. Consider the linear functional $h_{k\ell}^*$ dual to $h_{k\ell}$, restricted to K . To show that $y_{ij}^2 \in I$ we must compute the number of $h_{k\ell}^*$, $k < \ell$, that do not vanish on y_{ij} . This number is 3 and hence $y_{ij}^{3-2+1} \in I$.

The ideal I is stable under the action of \mathfrak{S}_n , and hence $(kn)y_{ij}^2 = (y_{ij} + y_{jk} + y_{kn})^2 \in I$. Subtracting off the quadratic terms proves that I contains $y_{ij}y_{ki} + y_{ji}y_{kj} + y_{ki}y_{jk}$. \square

Proposition 6.5. *The set of polynomials,*

$$y_{ij}^2, \quad (1 \leq i < j \leq n-1), \quad \underline{-y_{ij}y_{ik} + y_{ij}y_{jk} - y_{ik}y_{jk}}, \quad (1 \leq i < j < k \leq n-1).$$

forms a Gröbner basis under any term order where the leading terms are underlined above.

Proof. We use the fact that syzygies of polynomials with relatively prime leading terms need not be computed [4, Exercise 15.20]. By symmetry, the computation reduces to the case when $n = 4$. This case is easily checked with a computer (e.g., using [9]). \square

Proof of Lemma 6.3. The ideal of leading terms of J is $\langle y_{ij}^2 : 1 \leq i < j \leq n-1 \rangle + \langle y_{ij}y_{jk} : 1 \leq i < j < k \leq n-1 \rangle$. It follows that a basis for $\text{Sym}(K)/J$ consists of square-free monomials in the y_{ij} that avoid $y_{ij}y_{ik}$, for $1 \leq i < j < k \leq n$. These monomials are in obvious bijection with the set of decreasing forests on $n-1$ vertices. These are forests with vertex set $[n-1]$ where, in every component, each path directed away from the largest vertex in that component decreases. Adding a vertex labeled n and connecting the largest vertex in each component to n , we obtain a bijection between decreasing forests on $n-1$ vertices, and decreasing trees on n vertices. By [15, Proposition 1.5.5], there are exactly $(n-1)!$ such trees.

It follows that the dimension of $\text{Sym}(K)/J$ as a vector space is $(n-1)!$. Since this is the dimension of $S_{\mathcal{A}^+,-2}$ (recall, its Hilbert series is $(1+q)(1+2q)\cdots(1+(n-2)q)$ and there is a containment between J and $I_{\mathcal{A}^+,-2}$, the two ideals are equal. \square

To prove Theorem 6.2 we will need a result of Sundaram which we recall here.

Proposition 6.6 (Sundaram [17, Lemma 3.1]). *Let W_n and V_n be (possibly virtual) representations of S_n , and let $\mathbf{1}^\perp$ denote the orthogonal complement of the trivial representation in \mathbf{C}^n . Then the following are equivalent:*

- (1) $W_n \otimes \mathbf{1}^\perp \approx V_n$.
- (2) $W_n + V_n \approx \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} W_n$.

Proof of Theorem 6.2. By Theorem 6.3, we have identified $S_{\mathcal{A}^\perp, -2}$ with the \mathfrak{S}_n module Mathieu studies in [10, Theorem 4.4]. We denote the top degree part of this \mathfrak{S}_n module by Q_n . In *loc. cit.* it is shown that there is an isomorphism of \mathfrak{S}_n -modules,

$$\left(\text{Res}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}}(Q_{n+1}) \right) \approx Q_n \otimes \mathbf{1}^\perp.$$

Using Theorem 6.1 and Proposition 6.6,

$$\text{Lie}_n \oplus Q_n \approx \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} \text{Lie}_{n-1}$$

This proves that $Q_n \approx \text{Ind}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} (\text{Lie}_{n-1}) - \text{Lie}_n$. □

7. THE INTERNAL ZONOTOPAL IDEAL

In this section we describe in coordinates the defining ideal of $S_{\mathcal{G}^\perp, -2}$. We will use this to prove representation stability in the sense of Church, Ellenberg and Farb and developed for wreath products with symmetric groups by Sam and Snowden [16] and Gan and Li [7].

7.1. Generators. Recall that $\mathcal{G}^\perp \subset K$, where K has coordinates y_{ij}^k with $1 \leq i < j \leq n$ and $0 \leq k \leq m-1$.

Theorem 7.1. *Let \mathcal{G} be the reflection arrangement of $W = G(m, 1, n)$ ($m \geq 1$). Then the ideal defining $S_{\mathcal{G}^\perp, -2}$ in $\text{Sym}(K)$ is the sum of*

$$J_1 = \langle y_{ij}^k y_{ij}^{k'} : 1 \leq i < j \leq n, 0 \leq k, k' \leq m \rangle$$

with the smallest W -stable ideal $J_2 \subset \text{Sym}(K)$ containing the single element $y_{ij}^0 y_{jk}^0 + y_{ik}^0 y_{kj}^0 + y_{ji}^0 y_{ik}^0$.

Proof. We let I denote the ideal of $S_{\mathcal{G}^\perp, -2}$. We first show that $J_1, J_2 \subset I$. Consider the element $y_{ij}^k = h_i - \omega^k h_j - h_{ij}^k \in K$. The hyperplanes in \mathcal{G}^\perp that do not contain this vector are h_i, h_j and h_{ij}^k , hence $(y_{ij}^k)^2 \in I$. Similarly, the number of hyperplanes in \mathcal{G}^\perp that do not contain $y_{ij}^k - y_{ij}^{k'} = (\omega^{k'} - \omega^k) h_j - h_{ij}^k + h_{ij}^{k'}$ is 3, hence $(y_{ij}^k - y_{ij}^{k'})^2 \in I$. This proves $J_1 \subset I$. Next we consider $y_{ij}^0 + y_{jk}^0 + y_{ki}^0 = -(h_{ij}^0 + h_{jk}^0 + h_{ki}^0)$, which is contained in all but 3 hyperplanes in \mathcal{G}^\perp , so its square is in I . This shows that $J_2 \subset I$.

We claim that $\text{Sym}(K)/(J_1 + J_2)$ has the same dimension as $\text{Sym}(K)/I$ as a vector space. The non-zero monomials in $\text{Sym}(K)/J_1$ are in obvious bijection with graphs on vertex set $[n]$ whose edges are labeled with the numbers $0, 1, \dots, k-1$. Suppose that we have a monomial in $\text{Sym}(K)/(J_1 + J_2)$ whose corresponding graph has a 3-cycle. We can use the W_n invariance of this quoient to assume the three cycle

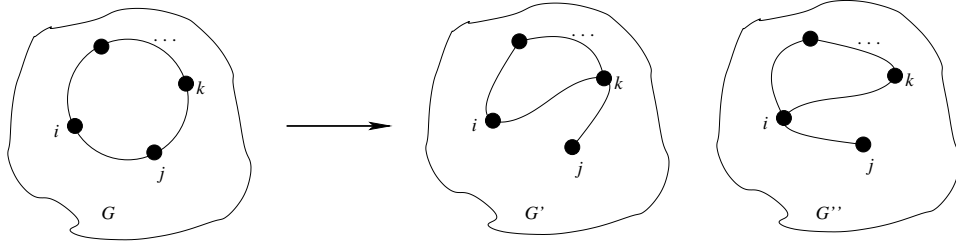


FIGURE 1. Rewriting a monomial corresponding to a graph with a cycle of length larger than 3 in terms of two monomials whose graphs have cycles of smaller length.

has two edges labeled 0. Rewriting the transformed monomial using the sole given generator of J_2 , we will obtain a sum of two monomials both of which reduce to zero modulo J_1 .

Suppose that we have a monomial whose corresponding graph has a cycle of length longer than 3. Similar to the argument above we can use the elements in J_2 to rewrite this monomial as a sum of monomials of graphs with shorter cycles. See Figure 1. The resulting sum then reduces to zero in the quotient by induction on the length of a cycle. We conclude that $Sym(K)/(J_1 + J_2)$ is spanned by monomials in bijection with forests on vertex set $[n]$ whose edges are labeled with the numbers $0, 1, \dots, k-1$.

Using the relations in J_2 , we see that out of the forests that span $Sym(K)/(J_1 + J_2)$, we can use only those whose underlying graph where each connected component is a path, and one of whose endpoints is labeled with the highest label in that component. The number of such paths with exactly one connected component is at once seen to be $m^{n-1}(n-1)!$, which is the dimension of the degree $n-1$ piece of $Sym(K)/I = S_{\mathcal{G}^\perp, -2}$. A calculation with the exponential formula shows that the number of such forests on n vertices is $1 \cdot (m+1) \cdot (2m+1) \cdots ((n-1)m+1)$, which is the dimension of $S_{\mathcal{G}^\perp, -2}$.

We conclude that $Sym(K)/I = Sym(K)/(J_1 + J_2)$, and hence $I = J_1 + J_2$, which is what we wanted to show. \square

7.2. Representation Stability. Recall that the group $W_n := G(m, 1, n)$ can be interpreted as the wreath product $\mathbf{Z}/m \wr \mathfrak{S}_n$. The category $\mathbf{FI}_{\mathbf{Z}/m}$ was introduced by Sam and Snowden in [16]. Its objects are finite sets and a map $R \rightarrow S$ between two sets is a pair (f, ρ) where $f : R \rightarrow S$ is an injection and $\rho : R \rightarrow \mathbf{Z}/m$. The composition of $(f, \rho) : R \rightarrow S$ with $(g, \sigma) : S \rightarrow T$ is defined by $(g \circ f, \tau)$ where $\tau(x) = \sigma(f(x))\rho(x)$.

A (complex) representation of $\mathbf{FI}_{\mathbf{Z}/m}$ is a sequence $M = (M_n)_{n \geq 0}$ of finite dimensional complex representations M_n of W_n , together with a sequence of compatible transition maps $M_n \rightarrow M_{n+1}$ that are W_n -equivariant.

Denote the reflection arrangement associated to the group W_n by \mathcal{G}_n , instead of the usual \mathcal{G} .

Proposition 7.2. *For each degree $k \geq 0$, the sequences of representations $((S_{\mathcal{G}_n^\perp, -2})_k)_{n \geq 0}$ and $((I_{\mathcal{G}_n^\perp, -2})_k)_{n \geq 0}$ are representations of the category $\mathbf{FI}_{\mathbf{Z}/m}$.*

Proof. Let K_n denote the ambient space of the arrangement \mathcal{G}_n^\perp . Then the obvious inclusion $K_n \rightarrow K_{n+1}$ is $G(m, 1, n)$ -equivariant. The corresponding injections $\text{Sym}(K_n) \rightarrow \text{Sym}(K_{n+1})$ are $G(m, 1, n)$ -equivariant. Let $I_n \subset \text{Sym}(K_n)$ denote the internal zonotopal ideal of \mathcal{G}_n^\perp . By Theorem 7.1 there is a $G(m, 1, n)$ -equivariant inclusion $I_n \rightarrow I_{n+1}$. It follows that the natural maps $\text{Sym}(K_n)/I_n \rightarrow \text{Sym}(K_{n+1})/I_{n+1}$ are W_n equivariant. Since all the maps are homogeneous, the result follows. \square

A representation M of $\text{FI}_{\mathbf{Z}/m}$ is said to be finitely generated if there are finitely many elements in the spaces M_j such that the smallest $\text{FI}_{\mathbf{Z}/m}$ submodule of M containing these elements is exactly M itself.

Proposition 7.3. *For each degree $k \geq 0$, the sequences of representations $((S_{\mathcal{G}_n^\perp, -2})_k)_{n \geq 0}$ and $((I_{\mathcal{G}_n^\perp, -2})_k)_{n \geq 0}$ are finitely generated.*

Proof. Theorem 7.1 shows finite generation of the graded pieces of the sequence of ideals directly. This also follows since the sequence $(\text{Sym}(K_n)_k)_n$ is finitely generated and the main theorem of [16] proves that subrepresentations of finitely generated objects are again finitely generated (this is certainly overkill). The graded pieces of the sequence $(S_{\mathcal{G}_n^\perp, -2})_n$ are finitely generated since they are quotients of a finitely generated object. \square

The importance of being finitely generated here is that it implies representation stability, as described in [7, Definition 1.10]. Recall that the irreducible (complex) representations of the wreath product $\mathbf{Z}/m \wr \mathfrak{S}_n$ are indexed by m -tuples of partitions $\underline{\lambda} = (\lambda^i)$ satisfying $|\underline{\lambda}| = \sum_{i=1}^m |\lambda^i| = n$. We denote this irreducible by $L(\underline{\lambda})$; we will not need the precise construction here. The entries of this m -tuple $\underline{\lambda}$ are in bijection with the irreducible characters of \mathbf{Z}/m and we assume that λ^1 corresponds to the trivial character. For n larger than $|\underline{\lambda}| + \lambda_1^1$ we let $\underline{\lambda}[n]$ denote the partition obtained by adding a single part of length $n - |\underline{\lambda}|$ to λ^1 . We say that a sequence $(M_n)_{n \geq 0}$ of W_n representations is representation stable if for all $n \gg 0$ the following conditions hold:

- (1) The map $M_n \rightarrow M_{n+1}$ is injective.
- (2) The image of M_n in M_{n+1} generates M_{n+1} as a W_{n+1} -module.
- (3) The irreducible decomposition is given by

$$M_n \approx \bigoplus_{\underline{\lambda}} L(\underline{\lambda}[n])^{\oplus m_{\underline{\lambda}}}$$

for some integers $m_{\underline{\lambda}} \geq 0$ that do not depend on n .

Gan and Li prove [7, Theorem 1.12] that $(M_n)_{n \geq 0}$ is finitely generated if and only if it is representation stable. Applying their result we have the following.

Theorem 7.4. *For each degree $k \geq 0$, the sequences of representations $((S_{\mathcal{G}_n^\perp, -2})_k)_{n \geq 0}$ and $((I_{\mathcal{G}_n^\perp, -2})_k)_{n \geq 0}$ are representation stable.*

7.3. Factorization of the regular representation. One might hope, based on what happens in type A, that $S_{\mathcal{G}_n^\perp, -2}$ could be made to carry an action of $G(m, 1, n+1)$. However, there does not appear to be an action generalizing (2). The goal of this section is to prove an analog of the result of Mathieu [10, Theorem 4.4] used in the proof of Theorem 6.2.

Proposition 7.5. *Let $E = E_0 \oplus E_1$ be the graded representation of $G(m, 1, n)$ that is trivial in degree 0 and equal to $\text{Ind}_{G(m, 1, n-1)}^{G(m, 1, n)} 1$ in degree 1. Let \mathcal{G}_n be the reflection arrangement of $G(m, 1, n)$. Then there is an isomorphism of graded $G(m, 1, n)$ -modules,*

$$\text{Res}_{G(m, 1, n)}^{G(m, 1, n+1)} S_{\mathcal{G}_{n+1}^\perp, -2} \approx S_{\mathcal{G}_n^\perp, -2} \otimes E.$$

Taking the top degree piece of both sides and applying Corollary 5.4 we obtain the following result.

Corollary 7.6. *Maintaining the notation of Theorem 7.5, there is an isomorphism of $G(m, 1, n)$ -modules,*

$$\mathbf{C}[G(m, 1, n)] \approx S_{\mathcal{G}_n^\perp, -2}(\text{top}) \otimes E_1.$$

We can unravel this using the proof of Sundaram's Proposition 6.6, which brought us to the Whitehouse representation in Theorem 6.2. We obtain a much more underwhelming result, since we arrive at the simple statement

$$\text{Ind}_{G(m, 1, n)}^{G(m, 1, n+1)} \mathbf{C}[G(m, 1, n)] \approx \mathbf{C}[G(m, 1, n+1)].$$

Proof of Proposition 7.5. From the proof of Theorem 7.1, a linear basis for $S_{\mathcal{G}_{n+1}^\perp, -2}$ consists of monomials indexed by certain forests on vertex set $[n+1]$. Specifically, the edges are labeled with $0, 1, \dots, m-1$, each component is a path, and the largest labeled vertex in each path has degree one. This means that there is a basis indexed by monomials that are either (a) not divisible by any variable of the form $y_{(n+1)i}^k$ or (b) divisible by exactly one variable of the form $y_{(n+1)i}^k$. This implies that there is an isomorphism of graded vector spaces,

$$S_{\mathcal{G}_n^\perp, -2} \otimes F \approx S_{\mathcal{G}_{n+1}^\perp, -2},$$

where $F = F_0 \oplus F_1$ is trivial in degree 0 and in degree 1 has basis $y_{(n+1)i}^k$, $1 \leq i \leq n$, $0 \leq k \leq m-1$. The multiplication map $S_{\mathcal{G}_n^\perp, -2} \otimes F \rightarrow S_{\mathcal{G}_{n+1}^\perp, -2}$ is visibly $G(m, 1, n)$ -equivariant.

The group $G(m, 1, n)$ acts by permuting the variables $y_{(n+1)i}^k$. The action is transitive on the set of variables. The stabilizer of $y_{(n+1)n}^0$ is precisely $G(m, 1, n-1)$ and this proves that $E_1 \approx F_1$ as representations of $G(m, 1, n)$. \square

8. TYPE B DECREASING TREES

In this section we compute a Gröbner basis of the ideal $I_{\mathcal{G}^\perp, -2}$ when $m = 2$, where W is the hyperoctohedral group. We use this to give a non-trivial generalization of the notion of decreasing trees, as discussed in Section 6.

We start with a definition. A \pm tree on n vertices is a rooted tree with vertex set $[n]$ and root vertex n , together with a $\{+, -\}$ -coloring of its edges. A decreasing \pm tree on n vertices is a \pm tree on n vertices such that on any path directed away from the root,

- (1) along every edge labeled $-$ the vertex labels decrease,

(2) for $i < j$ and arbitrary k , there is no path of the form

$$j \xrightarrow{-} i \xrightarrow{+} k ,$$

(3) for all $i_1 < i_2 < i_3 < i_4$, there is no subpath of any of the three forms, or their reverse,

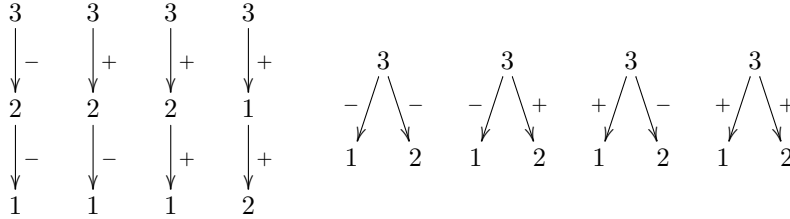
$$i_4 \xrightarrow{+} i_1 \xrightarrow{+} i_2 \xrightarrow{+} i_3 ,$$

$$i_4 \xrightarrow{+} i_1 \xrightarrow{+} i_3 \xrightarrow{+} i_2 ,$$

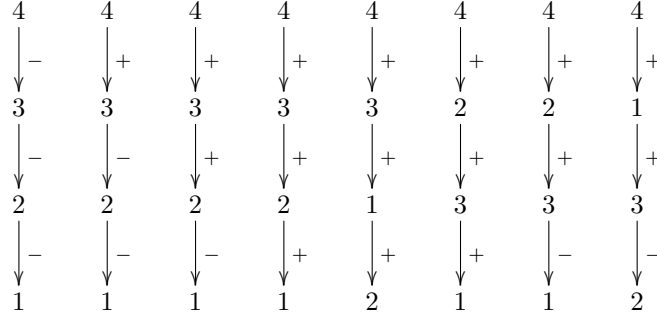
$$i_4 \xrightarrow{+} i_2 \xrightarrow{+} i_1 \xrightarrow{+} i_3 .$$

If we ignore the possibility of edges labeled $+$ this definition reduces to the usual definition of decreasing trees.

Example 8.1. Here are the 8 decreasing \pm trees on 3 vertices.



Example 8.2. There are 48 decreasing \pm trees on 4 vertices. Exactly 8 such trees are isomorphic to a path rooted at an endpoint, and these are displayed below.



Theorem 8.3. There are $2^{n-1}(n-1)!$ decreasing \pm trees on n vertices.

Proof. We know from the proof of Theorem 7.1 that the graphs corresponding to monomials not in $I_{\mathcal{G}^\perp, -2}$ are forests on $[n]$ whose edges are labeled 0 and 1. We change every edge labeled 0 to an edge labeled $-$ likewise with 1 and $+$. We now show that the monomials of decreasing \pm trees form a basis for the quotient $S_{\mathcal{G}^\perp, -2}(\text{top})$.

For this, we claim that a Gröbner basis of $I_{\mathcal{G}^\perp, -2}$ is furnished by the following monomials:

$$(y_{ij}^0)^2, \quad (y_{ij}^1)^2, \quad y_{ij}^0 y_{ij}^1, \quad y_{ij}^1 y_{jk}^1 y_{ki}^1.$$

where i, j and k range of distinct triples of integers, as well as the polynomials

$$\underline{y_{ji}^0 y_{ik}^0} + y_{ij}^0 y_{jk}^0 + y_{ik}^0 y_{kj}^0, \quad \underline{y_{ji}^1 y_{ik}^0} + y_{ij}^1 y_{jk}^1 - y_{ik}^0 y_{kj}^1,$$

where the indices range over distinct tuples of integers, and finally given $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ the polynomials,

$$\begin{aligned} & y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_2 i_4}^1 - y_{i_1 i_2}^1 y_{i_1 i_3}^1 y_{i_3 i_4}^1 - y_{i_1 i_2}^1 y_{i_2 i_4}^1 y_{i_3 i_4}^1 + \underline{y_{i_1 i_3}^1 y_{i_2 i_4}^1 y_{i_3 i_4}^1}, \\ & y_{i_1 i_3}^1 y_{i_2 i_3}^1 y_{i_1 i_4}^1 - y_{i_1 i_3}^1 y_{i_2 i_3}^1 y_{i_2 i_4}^1 - y_{i_1 i_3}^1 y_{i_1 i_4}^1 y_{i_2 i_4}^1 + \underline{y_{i_2 i_3}^1 y_{i_1 i_4}^1 y_{i_2 i_4}^1}, \\ & y_{i_1 i_2}^1 y_{i_2 i_3}^1 y_{i_1 i_4}^1 - y_{i_1 i_2}^1 y_{i_2 i_3}^1 y_{i_3 i_4}^1 - y_{i_1 i_2}^1 y_{i_1 i_4}^1 y_{i_3 i_4}^1 + \underline{y_{i_2 i_3}^1 y_{i_1 i_4}^1 y_{i_3 i_4}^1}. \end{aligned}$$

We use any term order where the underlined monomials are leading terms (such as graded-reverse lexicographic order with the y^0 's coming before the y^1 's).

To see this one applies the Buchberger algorithm to the generators of $I_{G^\perp, -2}$ given in Theorem 7.1. By [4, Exercise 15.20] we can reduce the computation to the case $n = 4$, since we need not resolve the syzygies of pair of polynomials with relatively prime leading terms. This case is, once again, handled at once by a computer to produce the described Gröbner basis. \square

A bijective proof of this result generalizing the one for ordinary decreasing trees is not known.

9. CONCLUSION

A natural question is whether Theorem 5.1 holds for all complex reflection groups in its present formulation. A number of problems can arise. A necessary condition is that the dimension of an induced character $\text{Ind}_C^W(\zeta)$ be equal to the dimension of $S_{G^\perp, -2}(\text{top})$, which is the product of the non-zero codegrees of W . For not well-generated reflection groups we take as an analog of a Coxeter element an element of W with order $h = d_n$. The equality of dimensions becomes,

$$d_1 d_2 \dots d_{n-1} = d_2^* d_3^* \dots d_n^*.$$

For the exceptional groups, we list the Shephard–Todd number of each group where $d_1 \dots d_{n-1} = d'_2 \dots d'_n$ (we have written the degrees and codegrees in weakly increasing order), which signifies that Theorem 5.1 could hold for this group: 5, 7, 10, 11, 18, 19, 26. This equality also holds for all of the groups $G(m, r, n)$, consisting of matrices in $G(m, 1, n)$ the product of whose non-zero entries is an m/r th root of unity.

In any groups where equality holds to would suffice to prove the following statement: *Let $\mathcal{G} \subset V$ be the reflection arrangement of a given complex reflection group W and let K be the kernel of the natural map $\mathbf{C}^{\mathcal{G}} \rightarrow V^*$. Then K has an eigenvector Y with eigenvalue $e^{2\pi i/h}$ whose W orbit spans K .*

In the case of $W = G(m, r, n)$ with $1 < r < n$ the reflection arrangement \mathcal{G} associated to W is the same as the arrangement associated to $G(m, 1, n)$. The proof of Lemma 5.2 shows that for $n \geq 4$, K contains an c -eigenvector Y whose W orbit include the single variable y_{12}^0 . The W orbit of y_{12}^0 does not span K , so more work is required.

Another natural question is whether one can give a more explicit (positive) formula for the graded pieces of $S_{\mathcal{G}^\perp, -2}$ other than its top. Already in type A this would be interesting.

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